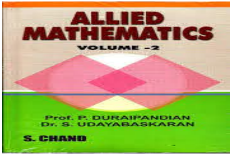
CAMA 15C: Mathematics - I
Unit-III Matrices



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Types of Problems

1. Symmetric and Skew-Symmetric Matrices
2. Orthogonal and unitary matrices
3. Rank of a matrix
4. Consistency of equations
5. Eigenvalues and eigenvectors
6. Cayley-Hamilton theorem verification and find inverse matrix

Matrix

A **matrix** is a rectangular arrangement of elements displayed in rows and columns put within a bracket ( ).

If a matrix A has m rows and n columns, then it is written as

$$A = (a_{ij}) \text{ where } 1 \leq i \leq m\,,\ 1 \leq j \leq n.$$

$$i.e., A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}_{m\times n}$$

The order of the matrix A is defined to be $m \times n$ (read as m by n).

Square matrix

A matrix in which number of rows is equal to the number of columns,

is called a **square matrix**.

That is, a matrix of order $n \times n$ is often referred to as a square matrix

of order n.

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{3\times 3}$$

Diagonal matrix

A square matrix $A = (a_{ij})_{n\times n}$ is called a **diagonal matrix**

if $a_{ij} = 0$ whenever $i \neq j$.

Example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}_{3\times 3}$$

Unit matrix

A square matrix in which all the diagonal entries are 1 and the rest are all zero is called a **unit matrix**.

Example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}$$

Transpose of a matrix

The **transpose** of a matrix is obtained by interchanging rows and columns of A and is denoted by A^T.

Example:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ a & 0 & 4 \\ b & c & 7 \end{pmatrix}_{3\times 3} \qquad A^T = \begin{pmatrix} 1 & a & b \\ -1 & 0 & c \\ 2 & 4 & 7 \end{pmatrix}_{3\times 3}$$

Properties

For any two matrices A and B of suitable orders, we have

(i) $(A^T)^T = A$

(ii) $(kA)^T = kA^T$ where k is any scalar

(iii) $(A \pm B)^T = A^T \pm B^T$

(iv) $(AB)^T = B^T A^T$

3.1 Symmetric and Skew-Symmetric Matrice

A square matrix A is said to be **symmetric** if $A^T = A$.

That is, $A = (a_{ij})_{n\times n}$ is a symmetric matrix, then $a_{ij} = a_{ji}$ for all i and j.

Example:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 4 \\ 2 & 4 & 7 \end{pmatrix}_{3\times 3} \qquad A^T = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 4 \\ 2 & 4 & 7 \end{pmatrix}_{3\times 3}$$

$$\Rightarrow A = A^T$$

Skew-symmetric

A square matrix A is said to be **skew-symmetric** if $A^T = -A$.

I.e., $A = (a_{ij})_{n\times n}$ is a symmetric matrix, then $a_{ij} = -a_{ji}$ for all i and j.

Example:

$$A = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix}_{3\times 3} \qquad -A^T = -\begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & -4 \\ 3 & 4 & 0 \end{pmatrix}_{3\times 3}$$

$$\Rightarrow A = -A^T$$

Note: A matrix which is both symmetric and skew-symmetric is a zero matrix

Results

1. For any square matrix A with real number entries, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew-symmetric matrix.
2. Any square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.
3. Let A and B be two symmetric matrices. AB is a symmetric matrix if and only if $AB = BA$.
4. (i) $AB + BA$ is a symmetric matrix.

   (ii) $AB - BA$ is a skew-symmetric matrix.

3.2. Orthogonal and unitary matrices

A matrix is said to be an **orthogonal matrix** if the product of a matrix

and its transpose gives an identity matrix.

I.e., $AA^T = A^T A = I$

Note:

A is orthogonal $\Leftrightarrow A^{-1} = A^T$.

Example:

Let $$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A\,A^T = \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example:

$$\text{Let} \quad A = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

$$A^T = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$$

$$A\,A^T = \begin{pmatrix} cos^2\theta + sin^2\theta & cos\theta sin\theta - sin\theta cos\theta \\ cos\theta sin\theta - sin\theta cos\theta & cos^2\theta + sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Conjugate matrix

A **conjugate matrix** of a matrix A is obtained by replacing each term with its complex conjugate.

It is denoted by $\overline{A}$

Example:

$$A = \begin{pmatrix} 1 & 1+i \\ 2-i & -3 \end{pmatrix} \qquad \overline{A} = \begin{pmatrix} 1 & 1-i \\ 2+i & -3 \end{pmatrix}$$

Properties:

For any two matrices A and B of suitable orders, we have

(i) $\overline{\overline{A}} = A$

(ii) $\overline{kA} = k\overline{A}$ where k is any scalar

(iii) $\overline{A+B} = \overline{A} + \overline{B}$

(iv) $\overline{AB} = \overline{A}\overline{B}$

Note: $\overline{A}^{T} = \overline{A^{T}} = A^{*}$

Unitary matrix

A square matrix is said to be the **unitary matrix** if

$$A\,A^* \;=\; A^*\,A \;=\; I$$

Note:

A is unitary $\Leftrightarrow A^{-1} \;=\; A^*$.

Example:

Let $$A = \begin{pmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{pmatrix}$$

$$A^T = \begin{pmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{-1+i}{2} & \frac{1-i}{2} \end{pmatrix}$$

$$\overline{A^T} = \begin{pmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{pmatrix}$$

$$AA^* = A\,\overline{A^T} = \begin{pmatrix} \frac{1+i}{2}\frac{1-i}{2} + \frac{-1+i}{2}\frac{-1-i}{2} & \frac{1+i}{2}\frac{1-i}{2} + \frac{-1+i}{2}\frac{1+i}{2} \\ \frac{1+i}{2}\frac{1-i}{2} + \frac{1-i}{2}\frac{-1-i}{2} & \frac{1+i}{2}\frac{1-i}{2} + \frac{1-i}{2}\frac{1+i}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1-i^2}{4} + \frac{1-i^2}{4} & \frac{1-i^2}{4} + \frac{i^2-1}{4} \\ \frac{1-i^2}{4} + \frac{i^2-1}{4} & \frac{1-i^2}{4} + \frac{1-i^2}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example:

Let $$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1+i \\ 1+i & 0 \end{pmatrix}$$

$$A^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1+i \\ -1+i & 0 \end{pmatrix}$$

$$\overline{A^T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1-i \\ -1-i & 0 \end{pmatrix}$$

$$AA^* = A\,\overline{A^T} = \frac{1}{2}\begin{pmatrix} 0+(-1+i)(-1-i) & 0 \\ 0 & (1+i)(1-i) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1-i^2 & 0 \\ 0 & 1-i^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Problems

1. Show that the matrix $\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$ is orthogonal.

2. Prove that the matrix $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is orthogonal.

3. Show that the matrix $A = \begin{pmatrix} \frac{1+i}{\sqrt{7}} & \frac{2+i}{\sqrt{7}} \\ \frac{2-i}{\sqrt{7}} & \frac{-1+i}{\sqrt{7}} \end{pmatrix}$ is unitary.