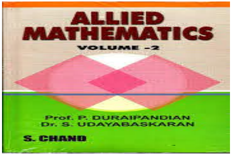
Allied Mathematics - I
Unit-II Theory of Equation



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Types of Problems

1. Relation between the roots and coefficient of equations.
2. Imaginary roots and irrational roots.
3. Transformation of equations.
4. Reciprocal equations.
5. Newton's method.

2.4 Reciprocal equations

Definition.

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Theorem 1.

A polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots. + a_2 x^2 + a_1 x + a_0 \ = \ 0, \ (a_n \neq 0)$$

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(i) $a_n = a_0 , \, a_{n-1} = a_1 , \, a_{n-2} = a_2 \ldots$

(ii) $a_n = -a_0 , \, a_{n-1} = -a_1 , \, a_{n-2} = -a_2 \ldots$

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For example, $6x^5 + x^4 - 43x^3 - 43x^2 + x + 6 = 0$ is of Type I.

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For example $6x^5 - 41x^4 + 97x^3 - 97x^2 + 41x - 6 = 0$ is of Type II.

Remark

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$P(x) = 0$ must be a reciprocal equation is not true.

For example, $2x^3 - 9x^2 + 12x - 4 = 0$ is a polynomial equation whose

roots are $2, 2, 1/2$.

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1. For an odd degree reciprocal equation of Type I,

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2. For an odd degree reciprocal equation of Type II,

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   solving this polynomial equation, we can get the roots of the given polynomial equation. (**Standard Type**)

Table

Types	Degree of $f(x)$	Sign of a_0 and a_n	Factor of $f(x)$
Type I	Even	Same	Solve
	Odd	Same	$x = -1$
Type II	Even	Opposite	$x = -1, 1$
	Odd	Opposite	$x = 1$

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$$4x^2 - 20x + 33 - \frac{20}{x} + \frac{4}{x^2} = 0. \qquad \text{(since } x \neq 0\text{)}$$

$$\therefore\ 4\left(x^2+\frac{1}{x^2}\right)-20\left(x+\frac{1}{x}\right)+33\,=\,0. \qquad \rightarrow (1)$$

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Let $y = x + \dfrac{1}{x}$. Then, we get

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Hence, the roots are $x \ = \ 2, \quad \frac{1}{2}, \quad 2, \quad \frac{1}{2}$.

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$$\therefore \left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0. \qquad \rightarrow (1)$$

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Substitute in (1), we get

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Substitute in (1), we get

$$(y^2 - 2) - 10y + 26 = 0$$

$$y^2 - 2 - 10y + 26 = 0$$

$y^2 - 10y + 24 = 0$

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$$(y-4)(y-6) = 0$$

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Case (i):

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$y = 6 \Rightarrow x + (1/x) = 6$

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$$\Rightarrow x = \frac{6 \pm \sqrt{36-4}}{2} \qquad \left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$$

$$x = \frac{6 \pm \sqrt{32}}{2}$$

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$$x = \frac{6 \pm \sqrt{16 \times 2}}{2}$$

$$x = \frac{6 \pm 4\sqrt{2}}{2}$$

$$x = \frac{6 \pm \sqrt{32}}{2}$$

$$x = \frac{6 \pm \sqrt{16 \times 2}}{2}$$

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Case (ii):

$$x = \frac{6 \pm \sqrt{32}}{2}$$

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Case (ii):

$$y = 4 \Rightarrow x + (1/x) = 4$$

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$$x = 3 \pm 2\sqrt{2}$$

Case (ii):

$y = 4 \Rightarrow x + (1/x) = 4$

$\Rightarrow x^2 + 1 = 4x$

$\Rightarrow x^2 - 4x + 1 = 0$

$$x = \frac{4 \pm \sqrt{16 - 4}}{2}$$

$$x = \frac{4 \pm \sqrt{16-4}}{2}$$
$$x = \frac{4 \pm \sqrt{12}}{2}$$

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$$x = \frac{4 \pm \sqrt{16 - 4}}{2}$$

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$$x = \frac{4 \pm \sqrt{4 \times 3}}{2}$$

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$$x = 2 \pm \sqrt{3}$$

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$$x = \frac{4 \pm \sqrt{4 \times 3}}{2}$$

$$x = \frac{4 \pm 2\sqrt{3}}{2}$$

$$x = 2 \pm \sqrt{3}$$

Hence, the roots are $x = 3 + 2\sqrt{2},\ 3 - 2\sqrt{2},\ 2 + \sqrt{3},\ 2 - \sqrt{3}$

Problem 3.

Solve the following equation $7x^3 - 43x^2 - 43x + 7 = 0$.

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This equation is **Type II odd degree Case 1** reciprocal equation.

Thus -1 is a solution and hence $x + 1$ is a factor.

Dividing the polynomial $7x^3 - 43x^2 - 43x + 7$ by the factor $x + 1$, we get

$$
\begin{array}{r|rrrr}
 & 7 & -43 & -43 & 7 \\
-1 & & -7 & 50 & -7 \\
\hline
 & 7 & -50 & 7 & 0
\end{array}
$$

$$\begin{array}{r|rrrr} & 7 & -43 & -43 & 7 \\ -1 & & -7 & 50 & -7 \\ \hline & 7 & -50 & 7 & 0 \end{array}$$

$\Rightarrow 7x^2 - 50x + 7 = 0 \qquad (ax^2 + bx + c = 0)$

$$\begin{array}{r|rrrr} & 7 & -43 & -43 & 7 \\ -1 & & -7 & 50 & -7 \\ \hline & 7 & -50 & 7 & 0 \end{array}$$

$\Rightarrow 7x^2 - 50x + 7 = 0$ $\qquad (\ ax^2 + bx + c = 0 \ )$

$$\Rightarrow x = \frac{50 \pm \sqrt{2500 - 196}}{14} \qquad \left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$$

$$\begin{array}{r|rrrr} & 7 & -43 & -43 & 7 \\ -1 & & -7 & 50 & -7 \\ \hline & 7 & -50 & 7 & 0 \end{array}$$

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$$\Rightarrow x = \frac{50 \pm \sqrt{2304}}{14}$$

$$\Rightarrow \; x = \frac{50 \pm 48}{14}$$

$$\Rightarrow \; x \; = \; \frac{50 \pm \; 48}{14}$$

$$\Rightarrow \; x \; = \; \frac{98}{14} \;\; , \; \frac{2}{14}$$

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$$\Rightarrow \; x = 7 \; , \; \frac{1}{7}$$

$$\Rightarrow \ x \ = \ \frac{50 \pm \ 48}{14}$$

$$\Rightarrow \ x \ = \ \frac{98}{14} \ , \ \frac{2}{14}$$

$$\Rightarrow \ x \ = 7 \ , \ \frac{1}{7}$$

Hence, the roots are $x \ = \ -1, \quad 7 \ , \ \frac{1}{7}$.

Problem 4.

Solve the following equation $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$.

Solution.

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This equation is **Type II odd degree Case 2** reciprocal equation.

Problem 4.

Solve the following equation $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$.

Solution.

Given $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$.

This equation is **Type II odd degree Case 2** reciprocal equation.

Thus 1 is a solution and hence $x - 1$ is a factor.

Problem 4.

Solve the following equation $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0.$

Solution.

Given $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0.$

This equation is **Type II odd degree Case 2** reciprocal equation.

Thus 1 is a solution and hence $x - 1$ is a factor.

Dividing the polynomial $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1$ by the factor

$x - 1$, we get

	1	−5	9	−9	5	−1
1		1	−4	5	−4	1
	1	−4	5	−4	1	0

$$\begin{array}{r|rrrrrr} & 1 & -5 & 9 & -9 & 5 & -1 \\ 1 & & 1 & -4 & 5 & -4 & 1 \\ \hline & 1 & -4 & 5 & -4 & 1 & 0 \end{array}$$

$\Rightarrow\ x^4 - 4x^3 + 5x^2 - 4x + 1 = 0.$ (**Type I - Standard**)

$$\begin{array}{c|cccccc} & 1 & -5 & 9 & -9 & 5 & -1 \\ 1 & & 1 & -4 & 5 & -4 & 1 \\ \hline & 1 & -4 & 5 & -4 & 1 & 0 \end{array}$$

$\Rightarrow x^4 - 4x^3 + 5x^2 - 4x + 1 = 0.$ (**Type I - Standard**)

$$x^2\left(x^2 - 4x + 5 - \frac{4}{x} + \frac{1}{x^2}\right) = 0.$$

$$\begin{array}{c|cccccc} & 1 & -5 & 9 & -9 & 5 & -1 \\ 1 & & 1 & -4 & 5 & -4 & 1 \\ \hline & 1 & -4 & 5 & -4 & 1 & 0 \end{array}$$

$$\Rightarrow x^4 - 4x^3 + 5x^2 - 4x + 1 = 0. \qquad (\textbf{ Type I - Standard})$$

$$x^2 \left(x^2 - 4x + 5 - \frac{4}{x} + \frac{1}{x^2} \right) = 0.$$

$$\Rightarrow x^2 - 4x + 5 - \frac{4}{x} + \frac{1}{x^2} = 0. \qquad (\text{since } x \neq 0)$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0. \qquad \rightarrow (1)$$

$$\Rightarrow \quad \left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0. \qquad \rightarrow (1)$$

Let $x + \frac{1}{x} = y$; and $x^2 + \left(\frac{1}{x}\right)^2 = y^2 - 2$.

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0. \qquad \rightarrow (1)$$

Let $x + \frac{1}{x} = y$; and $x^2 + \left(\frac{1}{x}\right)^2 = y^2 - 2\,.$

Then, we get

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0. \qquad \rightarrow (1)$$

Let $x + \dfrac{1}{x} = y$; and $x^2 + \left(\dfrac{1}{x}\right)^2 = y^2 - 2\,.$

Then, we get

$$(y^2 - 2) - 4y + 5 = 0.$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0. \qquad \rightarrow (1)$$

Let $x + \dfrac{1}{x} = y$; and $x^2 + \left(\dfrac{1}{x}\right)^2 = y^2 - 2$.

Then, we get

$$(y^2 - 2) - 4y + 5 = 0.$$

$$\Rightarrow y^2 - 4y + 3 = 0.$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0. \qquad \rightarrow (1)$$

Let $x + \frac{1}{x} = y$; and $x^2 + \left(\frac{1}{x}\right)^2 = y^2 - 2$.

Then, we get

$$(y^2 - 2) - 4y + 5 = 0.$$

$$\Rightarrow y^2 - 4y + 3 = 0.$$

$$\Rightarrow (y - 1)(y - 3) = 0.$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0. \qquad \rightarrow (1)$$

Let $x + \dfrac{1}{x} = y$; and $x^2 + \left(\dfrac{1}{x}\right)^2 = y^2 - 2$.

Then, we get

$$(y^2 - 2) - 4y + 5 = 0.$$

$$\Rightarrow y^2 - 4y + 3 = 0.$$

$$\Rightarrow (y-1)(y-3) = 0.$$

$$\Rightarrow y = 1, 3.$$

$y = 3 \Rightarrow x + (1/x) = 3$

$y = 3 \Rightarrow x + (1/x) = 3$

$\Rightarrow x^2 + 1 = 3x$

$y = 3 \Rightarrow x + (1/x) = 3$

$\Rightarrow x^2 + 1 = 3x$

$\Rightarrow x^2 - 3x + 1 = 0 \qquad (ax^2 + bx + c = 0)$

$y = 3 \Rightarrow x + (1/x) = 3$

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$\Rightarrow x^2 - 3x + 1 = 0 \qquad (\ ax^2 + bx + c = 0\ )$

$$\Rightarrow x = \frac{3 \pm \sqrt{9-4}}{2} \qquad \left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$$

$y = 3 \Rightarrow x + (1/x) = 3$

$\Rightarrow x^2 + 1 = 3x$

$\Rightarrow x^2 - 3x + 1 = 0 \qquad (ax^2 + bx + c = 0)$

$$\Rightarrow x = \frac{3 \pm \sqrt{9-4}}{2} \qquad \left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$$

$$\Rightarrow x = \frac{3 \pm \sqrt{5}}{2}$$

$y = 3 \Rightarrow \; x + (1/x) = 3$

$\Rightarrow \; x^2 + 1 = 3x$

$\Rightarrow \; x^2 - 3x + 1 = 0 \qquad\qquad (\; ax^2 + bx + c = 0 \;)$

$$\Rightarrow \; x \; = \; \frac{3 \pm \sqrt{9-4}}{2} \qquad\qquad \left(\; x \; = \; \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \; \right)$$

$$\Rightarrow \; x \; = \; \frac{3 \pm \sqrt{5}}{2}$$

$y = 1 \Rightarrow$ There exists no solution.

$y = 3 \Rightarrow x + (1/x) = 3$

$\Rightarrow x^2 + 1 = 3x$

$\Rightarrow x^2 - 3x + 1 = 0 \qquad (ax^2 + bx + c = 0)$

$$\Rightarrow x = \frac{3 \pm \sqrt{9-4}}{2} \qquad \left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$$

$$\Rightarrow x = \frac{3 \pm \sqrt{5}}{2}$$

$y = 1 \Rightarrow$ There exists no solution.

Hence, the roots are $x = 1, \ \frac{3+\sqrt{5}}{2}, \ \frac{3-\sqrt{5}}{2}$.

Problem 5.

Solve the following equation $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$.

Problem 6.

Solve the following equation $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$.

Problem 7.

Solve the following equation $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$.

Problem 8.

Solve the following equation $x^5 + 8x^4 + 21x^3 + 21x^2 + 8x + 1 = 0$.