Allied Mathematics - I

Unit-I ALGEBRA

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Types

1. Partial Fractions
2. Binomial Series
3. Exponential Series
4. Logarithmic Series

A function which is the sum of positive integral powers of a variable, say x, is called a **polynomial** in x.

Polynomial	**Degree**
$a_0x + a_1$	1
$a_0x^2 + a_1x + a_2$	2
$a_0x^3 + a_1x^2 + a_2x + a_3$	3

A rational expression of x is defined as the ratio of two polynomials in x, say $P(x)$ and $Q(x)$ where $Q(x) \neq 0$.

$$\frac{P(x)}{Q(x)} = \frac{a\ polynomial}{a\ polynomial}$$

Examples.

1. $\dfrac{x+3}{(x-1)(x+1)}$
2. $\dfrac{x^3+3x+1}{(x+2)(x^2+2x+1)}$

A rational expression $\dfrac{P(x)}{Q(x)}$ is called a **proper fraction**

if the degree of $P(x)$ is less than the degree of $Q(x)$.

Example.

$$\frac{x+3}{(x-1)(x+1)}$$

(Here degree of $P(x) = 1$ and degree of $Q(x) = 2$)

Improper fraction

A rational expression $\dfrac{P(x)}{Q(x)}$ is called a **improper fraction**

if the degree of $P(x)$ is equal to or larger than the degree of $Q(x)$.

Example.

$$\frac{x^3 + 3x + 1}{(x+2)(x^2+2x+1)}$$

(Here degree of $P(x) = 3$ and degree of $Q(x) = 3$)

Note.

Every improper fraction can be converted into a polynomial and a

Proper fraction

Types in PPF

There are **three** types in proper partial fraction.

1. Linear factors, no factor is repeated.

$$\frac{P(x)}{(x+a)(x+b)(x-c)} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x-c)}$$

2. Linear factors, Some of the factors are repeated.

$$\frac{P(x)}{(x+a)^3} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3}$$

3. Quadratic equation. $\dfrac{P(x)}{(ax^2+bx+c)} = \dfrac{(Ax+B)}{(ax^2+bx+c)}$

Problem 1. Resolve into partial fraction $\dfrac{x}{(x-1)(x+2)}$

Solution:

Let $\dfrac{x}{(x-1)(x+2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+2)}$

$$x = A(x+2) + B(x-1)$$

$$\text{Put } x = 1 \implies 1 = A(1+2) + B(1-1)$$

$$\implies 1 = 3A$$

$$\implies A = \tfrac{1}{3}$$

$$\text{Put } x = -2 \implies -2 = A(-2+2) + B(-2-1)$$

$$\implies -2 = -3B$$

$$\implies B = \tfrac{2}{3}$$

Thus, $$\frac{x}{(x-1)(x+2)} = \frac{1/3}{(x-1)} + \frac{2/3}{(x+2)}$$

$$\frac{x}{(x-1)(x+2)} = \frac{1}{3(x-1)} + \frac{2}{3(x+2)}$$

Problem 2. Resolve into partial fraction $\dfrac{2x+3}{(x-1)(x-2)(x-3)}$

Solution:

Let $\dfrac{2x+3}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-3)}$

$$2x+3 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$$\text{Put } x=1 \implies 5 = A(1-2)(1-3)$$

$$\implies 5 = A(-1)(-2)$$

$$\implies 5 = 2A$$

$$\implies A = \tfrac{5}{2}$$

$$\text{Put } x = 2 \implies 7 = B(2-1)(2-3)$$

$$\implies 7 = B(1)(-1)$$

$$\implies 7 = -B$$

$$\implies B = -7$$

$$\text{Put } x = 3 \implies 9 = C(3-1)(3-2)$$

$$\implies 9 = C(2)(1) \qquad \implies 9 = 2C$$

$$\implies C = \frac{9}{2}$$

Thus, $$\frac{2x+3}{(x-1)(x-2)(x-3)} = \frac{5/2}{(x-1)} + \frac{-7}{(x-2)} + \frac{9/2}{(x-3)}$$

Problem 3. Resolve into partial fraction $\dfrac{x}{(x-1)(2x+1)}$

Problem 4. Resolve into partial fraction $\dfrac{1}{(x)(x+1)(2x-1)}$

Problem 5. Resolve into partial fraction $\dfrac{x}{(x+3)(x-4)}$

Case 2. Linear factors (repeated)

Problem 6. Resolve into partial fraction $\dfrac{x+12}{(x-2)(x+1)^2}$

Solution:

Let $$\frac{x+12}{(x+1)^2(x-2)} = \frac{A}{(x-2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$x+12 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$$

$$\text{Put } x = 2 \implies 2+12 = A(3)^2$$

$$\implies A = 14/9$$

$$\text{Put } x = -1 \implies -1 + 12 = C(-1-2)$$

$$\implies C = -11/3$$

$$\text{Put } x = 0$$

$$\implies 0 + 12 = A(0+1)^2 + B(0-2)(0+1) + C(0-2)$$

$$\implies 12 = A - 2B - 2C$$

$$\implies 12 = 14/9 - 2B - 2(-11/3)$$

$$\implies 14/9 - 2B + 22/3 = 12$$

$$\implies (14+66)/9 - 2B = 12$$

$$\implies -2B = 12 - \tfrac{80}{9}$$

$$\implies -2B = \frac{108-80}{9}$$

$$\implies -2B = \frac{28}{9}$$

$$\implies B = -\frac{14}{9}$$

Thus, $$\frac{x+12}{(x+1)^2(x-2)} = \frac{14}{9(x-2)} - \frac{14}{9(x+1)} - \frac{11}{3(x+1)^2}$$

Problem 7. Resolve into partial fraction $\dfrac{x^2+1}{(2x-3)(x-1)^2}$

Solution:

$$\text{Let} \quad \frac{x^2+1}{(2x-3)(x-1)^2} = \frac{A}{(2x-3)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$x^2+1 = A(x-1)^2 + B(2x-3)(x-1) + C(2x-3)$$

$$\text{Put } x=1 \implies 1+1 = C\,(2-3)$$

$$\implies C = -2$$

$$
\begin{aligned}
\text{Put } x = \tfrac{3}{2} \implies (\tfrac{3}{2})^2 + 1 \; &= \; A\,(\tfrac{3}{2} - 1)^2 \\
\implies \tfrac{9}{4} + 1 \; &= \; A\,(\tfrac{3-2}{2})^2 \\
\implies \tfrac{13}{4} \; &= \; A\,\tfrac{1}{4} \\
\implies A \; &= \; 13
\end{aligned}
$$

Now, equating the coefficient of x^2, we get

$$1 = A + 2B$$

$$\implies 1 = 13 + 2B$$

$$\implies 2B = -12$$

$$\implies B = -6$$

Thus, $$\frac{x^2+1}{(2x-3)(x-1)^2} = \frac{13}{(2x-3)} - \frac{6}{(x-1)} - \frac{2}{(x-1)^2}$$

Problem 8. Resolve into partial fraction $\dfrac{3x^2+5}{(x-1)(x+1)^2}$

Problem 9. Resolve into partial fraction $\dfrac{5x^2-2x+60}{(x+5)(x-3)^2}$

Problem 10. Resolve into partial fraction $\dfrac{x+1}{x^2(x-1)}$

Problem 11. Resolve into partial fraction $\dfrac{2x+3}{(x^2+1)\,(x+4)}$

Solution:

Let $$\frac{2x+3}{(x^2+1)(x+4)} = \frac{A}{(x+4)} + \frac{(Bx+C)}{(x^2+1)}$$

$$2x+3 = A(x^2+1) + (Bx+C)(x+4)$$

$$\text{Put } x=-4 \implies -8+3 = A((-4)^2+1)$$

$$\implies -5 = 17A$$

$$\implies A = -\tfrac{5}{17}$$

$$
\begin{aligned}
\text{Put } x = 0 \implies 3 &= A + 4C \\
\implies 3 &= -\frac{5}{17} + 4C \\
\implies 3 + \frac{5}{17} &= 4C \\
\implies 4C &= \frac{56}{17} \\
\implies C &= \frac{56}{68} \\
\implies C &= \frac{14}{17}
\end{aligned}
$$

Now, equating the coefficient of x^2, we get

$$\implies A + B = 0$$

$$\implies -\tfrac{5}{17} + B = 0$$

$$\implies B = \tfrac{5}{17}$$

Thus, $\dfrac{2x+3}{(x^2+1)(x+4)} = -\dfrac{5}{17(x+4)} + \dfrac{5x+14}{17(x^2+1)}$

Problem 12. Resolve into partial fraction $\dfrac{2x+1}{(x^2+1)(x-1)}$

Solution:

Let $$\frac{2x+1}{(x^2+1)(x-1)} = \frac{A}{(x-1)} + \frac{(Bx+C)}{(x^2+1)}$$

$$2x+1 = A(x^2+1) + (Bx+C)(x-1)$$

$$\text{Put } x = 1 \implies 3 = 2A$$

$$\implies A = 3/2$$

$$\text{Put } x = 0 \implies A - C = 1$$

$$\implies 3/2 - C = 1$$

$$\implies -C = 1 - \frac{3}{2} \implies C = \frac{1}{2}$$

Now, equating the coefficient of x^2, we get $A + B = 0$

$$\implies 3/2 + B = 0$$

$$\implies B = -3/2$$

Thus, $$\frac{2x+1}{(x^2+1)(x-1)} = \frac{3}{2(x-1)} + \frac{(-3x+1)}{2(x^2+1)}$$

Problem 13. Resolve into partial fraction $\dfrac{x^2+x+1}{(x^2-1)(x+3)}$

Problem 14. Resolve into partial fraction $\dfrac{x^3-19x-15}{(x^2+1)(x+2)^3}$

Problem 15. Resolve into partial fraction

$$\frac{x^2}{(x^2+1)(x^2+2)(x^2+3)}$$

$$n! = n(n-1)(n-2)\ldots 3.2.1 \qquad 0! = 1.$$

$$^nC_r = \frac{n(n-1)(n-2)\ldots(n-(r-1))}{r(r-1)(r-2)\ldots 1} = \frac{n!}{(n-r)!r!}.$$

Example:

$$^5C_3 = \frac{5!}{(5-3)!3!} = \frac{5.4.3.2.1}{(2.1)\,3.2.1} = 10$$

$$^7C_4 = \frac{7!}{(7-4)!4!} = \frac{7.6.5.4.3.2.1}{(3.2.1)\,4.3.2.1} = 35$$

Pascal triangle is

			${}^0C_0$			
		${}^1C_0$		${}^1C_1$		
	${}^2C_0$		${}^2C_1$		${}^2C_2$	
${}^3C_0$		${}^3C_1$		${}^3C_2$		${}^3C_3$

			1			
		1		1		
	1		2		1	
1		3		3		1

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

[$n = 5 \quad \Longrightarrow \quad a^5 \quad a^4b \quad a^3b^2 \quad a^2b^3 \quad ab^4 \quad b^5$

Coefficient $\Longrightarrow \quad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$

$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5$]

1. Positive integral index

If n is any positive integer, then

$$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, a^0 b^n.$$

For any rational number n,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

for all real numbers x satisfying $|x| < 1$. $\quad (-1 < x < 1)$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots$$

Let us list some of them:

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$
2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$
3. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$
4. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots$

All the above expansions are valid only when $|x| < 1$.

$$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p-q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p-q)(p-2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots$$

$$(1-x)^{\frac{p}{q}} = 1 - \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p-q)}{2!}\left(\frac{x}{q}\right)^2 - \frac{p(p-q)(p-2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots$$

$$(1+x)^{-\frac{p}{q}} = 1 - \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 - \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots$$

$$(1-x)^{-\frac{p}{q}} = 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots$$

Problem 1. Find the sum to infinity of the series

$$1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \dots$$

Solution:

$$\text{Let } S = 1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \dots$$

$$= 1 + \frac{1}{(1 \times 3)} + \frac{1.3}{(1 \times 3)(2 \times 3)} + \frac{1.3.5}{(1 \times 3)(2 \times 3)(3 \times 3)} + \dots$$

$$= 1 + \frac{1}{1}\frac{1}{3} + \frac{1.3}{(1 \times 2)}\frac{1}{(3 \times 3)} + \frac{1.3.5}{(1 \times 2 \times 3)}\frac{1}{(3 \times 3 \times 3)} + \dots$$

$$
\begin{aligned}
S &= 1 + \frac{1}{1!}\frac{1}{3} + \frac{1.3}{2!}\frac{1}{(3\times 3)} + \frac{1.3.5}{3!}\frac{1}{(3\times 3\times 3)} + \dots \\
&= 1 + \frac{1}{1!}\left(\frac{1}{3}\right) + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3 + \dots \\
&= 1 + \frac{1}{1!}\left(\frac{1}{3}\right) + \frac{1(1+2)}{2!}\left(\frac{1}{3}\right)^2 + \frac{1(1+2)(1+2.2)}{3!}\left(\frac{1}{3}\right)^3 + \dots
\end{aligned}
$$

Here $p = 1$ (First term) $\quad$ (Nr term $1, 3, 5, 7, 9, \dots$)

$q = 2$ (Common difference)

$$\frac{x}{q} = \frac{1}{3}$$

$$\implies x = \frac{q}{3}$$

$$\implies x = \frac{2}{3}$$

Therefore, $S = (1-x)^{-p/q}$

$$\implies S = \left(1 - \tfrac{2}{3}\right)^{-1/2}$$

$$\implies S = \left(\tfrac{1}{3}\right)^{-1/2}$$

$$\therefore S = (3)^{1/2} = \sqrt{3}.$$

Problem 2. Find the sum to infinity of the series

$$\frac{1}{10} + \frac{1.4}{10.20} + \frac{1.4.7}{10.20.30} + \dots$$

Solution:

$$\text{Let } S = \frac{1}{10} + \frac{1.4}{10.20} + \frac{1.4.7}{10.20.30} + \dots$$

$$S + 1 = 1 + \frac{1}{10} + \frac{1.4}{10.20} + \frac{1.4.7}{10.20.30} + \dots$$

$$S + 1 = 1 + \frac{1}{(1\times10)} + \frac{1.4}{(1\times10)(2\times10)} + \frac{1.4.7}{(1\times10)(2\times10)(3\times10)} + \dots$$

$$S + 1 = 1 + \frac{1}{1}\left(\frac{1}{10}\right) + \frac{1.4}{1 \times 2}\left(\frac{1}{10}\right)^2 + \frac{1.4.7}{1 \times 2 \times 3}\left(\frac{1}{10}\right)^3 + \dots$$

$$S + 1 = 1 + \frac{1}{1!}\left(\frac{1}{10}\right) + \frac{1.4}{2!}\left(\frac{1}{10}\right)^2 + \frac{1.4.7}{3!}\left(\frac{1}{10}\right)^3 + \dots$$

Here $p = 1$ $(1, 4, 7, 10, \dots)$

$q = 3$

$$\frac{x}{q} = \frac{1}{10} \implies x = \frac{q}{10} \implies x = \frac{3}{10}$$

Therefore, $S + 1 = (1 - x)^{-p/q}$

$$S + 1 = \left(1 - \frac{3}{10}\right)^{-1/3}$$

$$S + 1 = \left(\frac{7}{10}\right)^{-1/3}$$

$$\therefore \quad S = \left(\frac{10}{7}\right)^{1/3} - 1$$

Problem 3. Find the sum to infinity of the series

$$\frac{1}{3.6} + \frac{1.3}{3.6.9} + \frac{1.3.5}{3.6.9.12} + \dots$$

Solution:

$$\text{Let} \quad S = \frac{1}{3.6} + \frac{1.3}{3.6.9} + \frac{1.3.5}{3.6.9.12} + \dots$$

$$(-1)S = \frac{(-1)1}{3.6} + \frac{(-1)1.3}{3.6.9} + \frac{(-1)1.3.5}{3.6.9.12} + \dots$$

$$-1S = \frac{(-1)1}{3.6} + \frac{(-1)1.3}{3.6.9} + \frac{(-1)1.3.5}{3.6.9.12} + \dots$$

$$-1S + 1 + \frac{(-1)}{3} = 1 + \frac{(-1)}{3} + \frac{(-1)1}{3.6} + \frac{(-1)1.3}{3.6.9} + \frac{(-1)1.3.5}{3.6.9.12} + \ldots$$

$$-S + 1 - \frac{1}{3} = 1 + \frac{(-1)}{3} + \frac{(-1)1}{3.6} + \frac{(-1)1.3}{3.6.9} + \frac{(-1)1.3.5}{3.6.9.12} + \ldots$$

$$-S + \frac{2}{3} = 1 + \frac{(-1)}{1!}\left(\frac{1}{3}\right) + \frac{(-1)1}{2!}\left(\frac{1}{3}\right)^2 + \ldots$$

Here $p = -1$ $\qquad \frac{x}{q} = 1/3$

$q = 2$ $\qquad x = \frac{2}{3}$

Therefore, $-S + \frac{2}{3} = (1 - x)^{-\frac{p}{q}}$

$$-S + \frac{2}{3} = \left(1 - \frac{2}{3}\right)^{1/2}$$

$$-S + \frac{2}{3} = \left(\frac{1}{3}\right)^{1/2}$$

$$-S = \left(\frac{1}{3}\right)^{1/2} - \frac{2}{3}$$

$$S = \frac{2}{3} - \left(\frac{1}{3}\right)^{1/2}$$

Problem 4. Find the sum to infinity of the series

$$\frac{11.14}{10.15.20} + \frac{11.14.17}{10.15.20.25} + \dots$$

Solution:

$$\text{Let} \quad S = \frac{11.14}{10.15.20} + \frac{11.14.17}{10.15.20.25} + \dots$$

$$S = \frac{11.14}{(2 \times 5)(3 \times 5)(4 \times 5)} + \frac{11.14.17}{(2 \times 5)(3 \times 5)(4 \times 5)(5 \times 5)} + \dots$$

$$\frac{S}{5} = \frac{11.14}{(1\times5)(2\times5)(3\times5)(4\times5)} + \frac{11.14.17}{(1\times5)(2\times5)(3\times5)(4\times5)(5\times5)} + \dots$$

$$\frac{S}{5} = \frac{11.14}{4!(5^4)} + \frac{11.14.17}{5!(5^5)} + \dots$$

$$\frac{5.8.S}{5} = \frac{5.8.11.14}{4!(5^4)} + \frac{5.8.11.14.17}{5!(5^5)} + \dots$$

$$\frac{5.8.S}{5} = \frac{5.8.11.14}{4!}\left(\frac{1}{5}\right)^4 + \frac{5.8.11.14.17}{5!}\left(\frac{1}{5}\right)^5 + \dots$$

$$8.S = 1 + \frac{5}{1!}\left(\frac{1}{5}\right) + \frac{5.8}{2!}\left(\frac{1}{5}\right)^2 + \frac{5.8.11}{3!}\left(\frac{1}{5}\right)^3 + \dots$$
$$- \left(1 + \frac{5}{1!}\left(\frac{1}{5}\right) + \frac{5.8}{2!}\left(\frac{1}{5}\right)^2 + \frac{5.8.11}{3!}\left(\frac{1}{5}\right)^3\right)$$

Here $p = 5; \quad q = 3; \quad \frac{x}{q} = \frac{1}{5} \implies x = 3/5$

$$8.S = (1-x)^{-p/q} - \left(1 + \tfrac{5}{1!}\left(\tfrac{1}{5}\right) + \tfrac{5.8}{2!}\left(\tfrac{1}{5}\right)^2 + \tfrac{5.8.11}{3!}\left(\tfrac{1}{5}\right)^3\right)$$

$$8.S = \left(1 - \frac{3}{5}\right)^{-5/3} - \left(1 + 1 + \frac{4}{5} + \frac{44}{75}\right)$$

$$8.S = \left(\frac{2}{5}\right)^{-5/3} - \left(2 + \frac{4}{5} + \frac{44}{75}\right)$$

$$8.S = \left(\frac{2}{5}\right)^{-5/3} - \left(\frac{150 + 60 + 44}{75}\right)$$

$$S = \frac{1}{8}\left[\left(\frac{5}{2}\right)^{5/3} - \left(\frac{254}{75}\right)\right]$$

Problem 5. Find the sum to infinity of the series

$$1 + \frac{3}{4} + \frac{3.5}{2.4^2} + \frac{3.5.7}{2.3.4^3} + \dots$$

Problem 6. Sum the series

$$\frac{1}{6} + \frac{1.4}{6.12} + \frac{1.4.7}{6.12.18} + \dots$$

Problem 7. Sum the series

$$\frac{5}{3.6} + \frac{5.7}{3.6.9} + \frac{5.7.9}{3.6.9.12} + \dots$$

Problem 8. Find the sum to infinity of the series

$$\frac{15}{16} + \frac{15.21}{16.24} + \frac{15.21.27}{16.24.32} + \dots$$

Problem 9. Sum the series

$$\frac{1}{24} - \frac{1.3}{24.32} + \frac{1.3.5}{24.32.40} - \dots$$

Problem 10. Sum the series

$$\frac{1}{9.18} - \frac{1.3}{9.18.27} + \frac{1.3.5}{9.18.27.36} + \dots$$

3. Exponential Series

For any real number x,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

i.e., $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Properties and values of e

1. $e^{x+y} = e^x e^y$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $e^{x-y} = \dfrac{e^x}{e^y}$
4. $e^{rx} = (e^x)^r \quad \forall r$, rational
5. $e^1 = 2.71828182845\ldots; \qquad 2 < e < 3$
6. $e^0 = 1$
7. $e^{\infty} = \infty$
8. $e^{-\infty} = 0$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$\frac{e^{x} + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$\frac{e^{x} - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

The nth term of the series $a, a+d, a+2d, a+3d, \dots$ is

$$T_n = a + (n-1)d$$

where **a** - first term and **d** - common difference

$$e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$$

$$\frac{e^1 + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots$$

$$\frac{e^1 - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \dots$$

$$\frac{Nr}{Dr} \implies Nr = A + B(Dr) + C(Dr)(Dr-1) + D(Dr)(Dr-1)(Dr-2) + \dots$$

Problem 1. Find the sum to infinity of the series

$$1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \dots$$

Solution:

$$\text{Let } S = 1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \dots$$

The *n*th term of the series is

$$T_n = \frac{n^3}{n!} \qquad (\, 1, 2, 3, 4, \dots)$$

$$S = \sum_{n=1}^{\infty} T_n$$

$$T_n = \frac{n^3}{n(n-1)!} \qquad (\text{Since } 5! = 5.4.3.2.1 = 5.(4.3.2.1) = 5(4!))$$

$$T_n = \frac{n^2}{(n-1)!}$$

Let $\quad n^2 = A + B(n-1) + C(n-1)(n-2)$

Put $n = 1 \implies A = 1$

$$n = 2 \implies A + B = 4$$

$$\implies 1 + B = 4$$

$$\implies B = 3$$

$$n = 0 \implies A - B + 2C = 0$$

$$\implies 1 - 3 + 2C = 0$$

$$\implies 2C = 2$$

$$\implies C = 1$$

Therefore, $n^2 = 1 + 3(n-1) + (n-1)(n-2)$

$$\text{Thus, } T_n = \frac{n^2}{(n-1)!} = \frac{1 + 3(n-1) + (n-1)(n-2)}{(n-1)!}$$

$$T_n = \frac{n^2}{(n-1)!} = \frac{1}{(n-1)!} + \frac{3(n-1)}{(n-1)!} + \frac{(n-1)(n-2)}{(n-1)!}$$

$$T_n = \frac{1}{(n-1)!} + \frac{3(n-1)}{(n-1)(n-2)!} + \frac{(n-1)(n-2)}{(n-1)(n-2)(n-3)!}$$

$$T_n = \frac{1}{(n-1)!} + \frac{3}{(n-2)!} + \frac{1}{(n-3)!}$$

Put $n = 1, 2, 3, \ldots$

$$T_1 = \frac{1}{0!}$$

$$T_2 = \frac{1}{1!} + \frac{3}{0!}$$

$$T_3 = \frac{1}{2!} + \frac{3}{1!} + \frac{1}{0!}$$

$$T_4 = \frac{1}{3!} + \frac{3}{2!} + \frac{1}{1!}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Since $S = \sum_{n=1}^{\infty} T_n$

$$S = \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\right) + 3\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots\right)$$

$$+ \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\right)$$

$$S = e + 3e + e$$

$$\therefore S = 5e$$

Problem 2. Find the sum to infinity of the series

$$\frac{1^2}{3!} + \frac{2^2}{5!} + \frac{3^2}{7!} + \ldots$$

Solution:

$$\text{Let } S = \frac{1^2}{3!} + \frac{2^2}{5!} + \frac{3^2}{7!} + \ldots$$

The *n*th term of the series is

$$T_n = \frac{n^2}{(2n+1)!}$$

$$(Since\ 3,\ 5,\ 7 \ldots \implies t_n = 3 + (n-1)2 = 2n+1)$$

Let $n^2 = A + B(2n+1) + C(2n+1)(2n)$

Put $n = -1/2 \implies A = 1/4$

Put $n = 0 \implies A + B = 0$

$\implies 1/4 + B = 0$

$\implies B = -1/4$

put $n = 1 \implies A + 3B + 6C = 1$

$\implies 1/4 - 3/4 + 6C = 1 \implies -1/2 + 6C = 1$

$\implies 6C = 1 + (1/2) \implies 6C = 3/2 \implies C = 1/4$

$$n^2 = 1/4 - (1/4)(2n+1) + (1/4)(2n+1)(2n)$$

$$T_n = \frac{n^2}{(2n+1)!} = \frac{1}{4}\frac{1}{(2n+1)!} - \frac{1}{4}\frac{2n+1}{(2n+1)!} + \frac{1}{4}\frac{(2n+1)(2n)}{(2n+1)!}$$

$$T_n = \frac{1}{4}\frac{1}{(2n+1)!} - \frac{1}{4}\frac{1}{(2n)!} + \frac{1}{4}\frac{1}{(2n-1)!}$$

Put $n = 1, 2, 3, \ldots$

$$T_1 = \frac{1}{4}\frac{1}{3!} - \frac{1}{4}\frac{1}{2!} + \frac{1}{4}\frac{1}{1!}$$

$$T_2 = \frac{1}{4}\frac{1}{5!} - \frac{1}{4}\frac{1}{4!} + \frac{1}{4}\frac{1}{3!}$$

$$T_3 = \frac{1}{4}\frac{1}{7!} - \frac{1}{4}\frac{1}{6!} + \frac{1}{4}\frac{1}{5!}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$Since\ S = \sum_{n=1}^{\infty} T_n$$

$$S = \frac{1}{4}\left(\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\dots\right) - \frac{1}{4}\left(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots\right)$$

$$+\frac{1}{4}\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\dots\right)$$

$$S = \frac{1}{4}\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\dots-1\right)$$

$$-\frac{1}{4}\left(\frac{1}{0!}+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots-1\right)$$

$$+\frac{1}{4}\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\dots\right)$$

$$S = \frac{1}{4}\left(\frac{e-e^{-1}}{2}-1\right) - \frac{1}{4}\left(\frac{e+e^{-1}}{2}-1\right) + \frac{1}{4}\left(\frac{e-e^{-1}}{2}\right)$$

$$S = \frac{1}{4}\left(\frac{e-e^{-1}-2}{2}\right) - \frac{1}{4}\left(\frac{e+e^{-1}-2}{2}\right) + \frac{1}{4}\left(\frac{e-e^{-1}}{2}\right)$$

$$S = \left(\frac{e-e^{-1}-2-(e+e^{-1}-2)+e+e^{-1}}{8}\right)$$

$$S = \left(\frac{e-e^{-1}}{8}\right) = \left(\frac{e-\frac{1}{e}}{8}\right)$$

$$\therefore S = \left(\frac{e^2-1}{8e}\right)$$

Problem 3. Find the sum to infinity of the series

$$\frac{2.3}{3!} + \frac{3.5}{4!} + \frac{4.7}{5!} + \dots$$

Solution:

Let $S = \frac{2.3}{3!} + \frac{3.5}{4!} + \frac{4.7}{5!} + \dots$

The nth term of the series is

$$T_n = \frac{(n+1)(2n+1)}{(n+2)!}$$

$$\left(\begin{array}{l} 2, 3, 4 \dots \implies t_n = 2 + (n-1) = n+1 \\ 3, 5, 7 \dots \implies t_n = 3 + (n-1)2 = 2n+1 \\ 3, 4, 5 \dots \implies t_n = 3 + (n-1) = n+2 \end{array} \right)$$

Let $(n+1)(2n+1) = A + B(n+2) + C(n+2)(n+1)$

Put $n = -2 \implies A = (-1)(-3) \implies A = 3$

Put $n = -1 \implies A + B = 0 \implies B = -3$

Equating the coefficient of n^2, $\implies C = 2$

$\therefore (n+1)(2n+1) = 3 - 3(n+2) + 2(n+2)(n+1)$

$$T_n = \frac{(n+1)(2n+1)}{(n+2)!} = \frac{3}{(n+2)!} - \frac{3(n+2)}{(n+2)!} + \frac{2(n+1)(n+2)}{(n+2)!}$$

$$T_n = \frac{3}{(n+2)!} - \frac{3}{(n+1)!} + \frac{2}{n!}$$

Put $n = 1, 2, 3, \ldots$

$$T_1 = 3\frac{1}{3!} - 3\frac{1}{2!} + 2\frac{1}{1!}$$

$$T_2 = 3\frac{1}{4!} - 3\frac{1}{3!} + 2\frac{1}{2!}$$

$$T_3 = 3\frac{1}{5!} - 3\frac{1}{4!} + 2\frac{1}{3!}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\text{Since } S = \sum_{n=1}^{\infty} T_n$$

$$S = 3\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots - 1 - \frac{1}{1!} - \frac{1}{2!}\right)$$

$$-3\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots - 1 - \frac{1}{1!}\right)$$

$$+2\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots - 1\right)$$

$$S = 3\left(e - 1 - \frac{1}{1!} - \frac{1}{2!}\right) - 3\left(e - 1 - \frac{1}{1!}\right) + 2(e - 1)$$

$$S = 3\left(e - 2 - \frac{1}{2!}\right) - 3(e - 2) + 2(e - 1)$$

$$S = 3\left(\frac{2e - 5}{2}\right) - 3e + 6 + 2e - 2$$

$$S = \left(\frac{6e - 15}{2}\right) - e + 4 = \left(\frac{6e - 15 - 2e + 8}{2}\right)$$

$$S = \left(\frac{4e - 7}{2}\right) = 2e - \frac{7}{2}$$

Problem 4. Find the sum to infinity of the series

$$\frac{1^2}{1!} + \frac{1^2+2^2}{2!} + \frac{1^2+2^2+3^2}{3!} + \dots$$

Problem 5. Find the sum to infinity of the series

$$5 + \frac{2.6}{1!} + \frac{3.7}{2!} + \frac{4.8}{3!} + \dots$$

Problem 6. Sum to infinity of the series

$$\sum_{n=0}^{\infty} \frac{5n+1}{(2n+1)!}$$

consider $f(x) = a^x, \; x \in \mathcal{R}$ for $0 < a \neq 1$.

Here $f(x) = a^x$ is called **exponential function with base** a.

$f(x) = e^x, \; x \in \mathcal{R}$ is the most important one as it has applications

in many areas like mathematics, science and economics.

Observed that $f(x)$ is a **bijection.** $(1 - 1 \,\&\,$ onto $)$

Hence it has an inverse.

We call this inverse function as **logarithmic function**.

It is denoted by $\log_a(...)$.

For $0 < a \neq 1$, we have $y = a^x$ is equivalent to $\log_a y = x$.

Note:

(i) $\log_a(...)$ defined only for positive real numbers.

(ii) $a^0 = 1$ for any base a and hence $\log_a(1) = 0$ for any base a.

Properties of Logarithm

(i) $a^{log_a x} = x$ for all $x \in (0, \infty)$ and $log_a(a^y) = y$ for all $y \in R$.

(ii) For any $x, y > 0$, $log_a(xy) = log_a x + log_a y$.

(Product Rule)

(iii) For any $x, y > 0$, $log_a(\frac{x}{y}) = log_a x - log_a y$.

(Quotient Rule)

(iv) For any $x > 0$ and $r \in R$, $log_a x^r = r\, log_a x$.

(Power Rule)

(v) For any $x > 0$, with a and b as bases,

$$log_b x = \frac{log_a x}{log a b}$$ (Change of base formula.)

Note:

(i) If $a = 10$, then the logarithmic function $\log_{10}x$ is called

the **common logarithm.**

(ii) If $a = e$, (an irrational number, approximately equal to 2.718),

then the logarithmic function $\log_e x$ is called

the **natural logarithm**.

It is denoted by $\ln x$.

$$\log\,(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \text{ when } |\,x\,| < 1$$

$$-\log\,(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$$

$$\log\,2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Problem 1.

Show that $\frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \ldots = \log 2 - \frac{1}{2}$

Solution:

Let $S = \frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \ldots$

The nth term of the series is

$$T_n = \frac{1}{(2n-1)(2n)(2n+1)}$$

(Since $t_n = a + (n-1)\,d$; a - first term d - comman difference
$1, 3, 5, \ldots$ $2, 4, 6, \ldots$ $3, 5, 7 \ldots$)

Consider

$$\frac{1}{(2n-1)(2n)(2n+1)} = \frac{A}{(2n-1)} + \frac{B}{(2n)} + \frac{C}{(2n+1)}$$

$$1 = A(2n)(2n+1) + B(2n-1)(2n+1) + C(2n-1)(2n)$$

Put $n = 0 \implies -B = 1 \implies B = -1$

Put $n = 1/2 \implies 2A = 1 \implies A = 1/2$

Put $n = -1/2 \implies 2C = 1 \implies C = 1/2$

$$T_n = \frac{1}{(2n-1)(2n)(2n+1)} = \frac{1/2}{(2n-1)} - \frac{1}{(2n)} + \frac{1/2}{(2n+1)}$$

Put $n = 1, 2, 3, \ldots$

$$T_1 = \frac{1/2}{1} - \frac{1}{(2)} + \frac{1/2}{3}$$

$$T_2 = \frac{1/2}{3} - \frac{1}{(4)} + \frac{1/2}{5}$$

$$T_3 = \frac{1/2}{5} - \frac{1}{(6)} + \frac{1/2}{7}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$S = \sum_{n=1}^{\infty} T_n$$

$$S = \frac{1/2}{1} + \left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots \right)$$

$$S = \frac{1}{2} + \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots - 1 \right)$$

$$S = \frac{1}{2} + (\log 2 - 1)$$

$$S = \log 2 - \frac{1}{2}$$

Problem 2.

Show that $\dfrac{1}{1.1.3} + \dfrac{1}{2.3.5} + \dfrac{1}{3.5.7} + \ldots = 2\log 2 - 1$

Solution:

Let $S = \dfrac{1}{1.1.3} + \dfrac{1}{2.3.5} + \dfrac{1}{3.5.7} + \ldots$

The nth term of the series is

$$T_n = \frac{1}{n(2n-1)(2n+1)}$$

(Since $t_n = a + (n-1)\,d$; a - first term d - comman difference

$1, 2, 3, \ldots \qquad 1, 3, 5, \ldots \qquad 3, 5, 7 \ldots)$

$$\therefore\ T_n = \frac{2}{(2n-1)(2n)(2n+1)}$$

Now, $$\frac{2}{(2n-1)(2n)(2n+1)} = \frac{A}{(2n-1)} + \frac{B}{(2n)} + \frac{C}{(2n+1)}$$

$$2 = A(2n)(2n+1) + B(2n-1)(2n+1) + C(2n-1)(2n)$$

Put $n = 0 \implies -B = 2 \implies B = -2$

Put $n = 1/2 \implies 2A = 2 \implies A = 1$

Put $n = -1/2 \implies 2C = 2 \implies C = 1$

$$T_n = \frac{2}{(2n-1)(2n)(2n+1)} = \frac{1}{(2n-1)} - \frac{2}{(2n)} + \frac{1}{(2n+1)}$$

Put $n = 1, 2, 3, \ldots$

$$T_1 = \frac{1}{1} - \frac{2}{2} + \frac{1}{3}$$

$$T_2 = \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$$

$$T_3 = \frac{1}{5} - \frac{2}{6} + \frac{1}{7}$$

$$\vdots \quad \vdots \quad \vdots$$

$$S = \sum_{n=1}^{\infty} T_n$$

$$S = 1 + \left(-\frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \dots \right)$$

$$S = 1 + 2 \left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots \right)$$

$$S = 1 + 2 \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots - 1 \right)$$

$$S = 1 + 2 \, (\, log \; 2 - 1 \,)$$

$$S = 2 \, log \; 2 - 1$$

Problem 3.

Show that

$$\left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)\frac{1}{9}+\left(\frac{1}{5}+\frac{1}{6}\right)\frac{1}{9^2}+\ldots = 2 - 2\log 2$$

Problem 4.

Show that $\dfrac{1}{1.3}+\dfrac{1}{2.5}+\dfrac{1}{3.7}+\ldots = 2 - 2\log 2$

Problem 5.

Show that

$$1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\ldots = \log\sqrt{12}$$